

Einstein-Podolsky-Rosen-like separability criteria for two-mode Gaussian states

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We investigate the separability of the two-mode Gaussian states by using the variances of a pair of Einstein-Podolsky-Rosen (EPR)-like observables. Our starting point is inspired by the general necessary condition of separability introduced by Duan *et al.* [Phys. Rev. Lett. **84**, 2722 (2000)]. We evaluate the minima of the normalized forms of both the product and sum of such variances, as well as that of a regularized sum. Making use of Simon's separability criterion [Phys. Rev. Lett. **84**, 2726 (2000)], we prove that these minima are separability indicators in their own right. By optimizing the EPR-like uncertainty defined in the letter of Duan *et al.*, we further derive a separability indicator for two-mode Gaussian states with no reference to the positivity of the partial transpose (PPT) of the density matrix. We prove that the corresponding EPR-like condition of separability is manifestly equivalent to Simon's PPT one. The consistency of these two distinct approaches (EPR-like and PPT) affords a better understanding of the examined separability problem, whose explicit solution found long ago by Simon covers all situations of interest.

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I. INTRODUCTION

Detecting and measuring quantum entanglement represent one of the goals of quantum information science. In the last two decades, a large amount of work has been invested in writing efficient separability criteria for both discrete- and continuous-variable systems. Although bipartite entanglement seems to be the simplest to detect and measure, practically we are still forced to apply different criteria when discussing it for mixed states of composite systems.

As shown by Peres [1], a necessary condition of separability for an arbitrary two-party state is the requirement to have a non-negative partially transposed density matrix. In the case of discrete-variable systems, this requirement of positive partial transposition (PPT) is also a sufficient condition of separability only for states on $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathbb{C}^3$ Hilbert spaces [2]. For bipartite states of continuous-variable systems, the PPT condition was first applied by Simon [3]. Specifically, Simon proved that preservation of the non-negativity of the density matrix under partial transposition is not only a necessary, but also a sufficient condition for the separability of two-mode Gaussian states (TMGSs). Moreover, the partial transposition criterion could be expressed in an elegant symplectically invariant form valid for any TMGS [3]. To detect the continuous-variable entanglement for non-Gaussian states, Shchukin and Vogel have derived an infinite series of inequalities for the moments of the state required by the PPT condition [4]. Similar inequalities were obtained in Refs. [5, 6].

A somewhat parallel method to get general inseparability criteria for two-mode states originates in a practical procedure proposed by Reid for demonstrating the Einstein-Podolsky-Rosen (EPR) paradox [7] in a non-degenerate parametric amplifier [8]. This was done by using two non-local observables linearly built with the canonical quadrature operators of the modes, \hat{q}_j, \hat{p}_j , ($j = 1, 2$) [8, 9]:

$$\hat{Q}(\lambda) := \hat{q}_1 - \lambda \hat{q}_2, \quad \hat{P}(\mu) := \hat{p}_1 + \mu \hat{p}_2, \quad (1.1)$$

where λ and μ are adjustable positive parameters. As a consequence of their commutation relation,

$$[\hat{Q}(\lambda), \hat{P}(\mu)] = i(1 - \lambda\mu)\hat{I}, \quad (1.2)$$

we get the weak (Heisenberg) form of the uncertainty principle,

$$\Delta Q(\lambda) \Delta P(\mu) \geq \frac{1}{2} |1 - \lambda\mu|, \quad (1.3)$$

which has to be fulfilled by any quantum state. In Eq. (1.3) and in the sequel as well, $(\Delta A)_{\hat{\rho}}$ denotes the standard deviation of the observable \hat{A} in the state $\hat{\rho}$, which is the square root of the variance

$$[(\Delta A)_{\hat{\rho}}]^2 := \left\langle \left(\hat{A} - \langle \hat{A} \rangle_{\hat{\rho}} \hat{I} \right)^2 \right\rangle_{\hat{\rho}} = \langle \hat{A}^2 \rangle_{\hat{\rho}} - \left(\langle \hat{A} \rangle_{\hat{\rho}} \right)^2.$$

Unless $\lambda = \mu = 1$, the operators (1.1) are not genuine EPR observables since they do not commute. In Refs. [8, 9], a possible experimental observation of the inequality

$$\Delta Q(\lambda) \Delta P(\mu) < \frac{1}{2} \quad (1.4)$$

is interpreted as a sufficient condition to detect an EPR paradox. It is interesting to remark that the EPR paradox [7] as shown in Eq. (1.4) and the concept of steering

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introduced by Schrödinger [10] as analyzed in Refs.[11–13] were proven to be equivalent descriptions of non-locality. Moreover, the EPR-steering turned out to be a different kind of non-locality stronger than quantum inseparability [11, 12].

An important piece of progress in studying separability starting with the uncertainty principle was made by Duan *et al.* [14]. They have introduced another family of EPR-like uncertainties in terms of the variances of the non-local one-parameter operators:

$$\begin{aligned}\hat{Q}(\alpha) &:= \alpha \hat{q}_1 - \frac{1}{\alpha} \hat{q}_2, & \hat{P}_{\pm}(\alpha) &:= \alpha \hat{p}_1 \pm \frac{1}{\alpha} \hat{p}_2, \\ (\alpha > 0).\end{aligned}\quad (1.5)$$

The commutation relations

$$[\hat{Q}(\alpha), \hat{P}_{\pm}(\alpha)] = i \left(\alpha^2 \mp \frac{1}{\alpha^2} \right) \hat{I} \quad (1.6)$$

lead to the product-form uncertainty relations for the EPR-like observables (1.5),

$$\Delta Q(\alpha) \Delta P_{\pm}(\alpha) \geq \frac{1}{2} \left| \alpha^2 \mp \frac{1}{\alpha^2} \right|, \quad (1.7)$$

which imply the sum-form inequalities:

$$[\Delta Q(\alpha)]^2 + [\Delta P_{\pm}(\alpha)]^2 \geq \left| \alpha^2 \mp \frac{1}{\alpha^2} \right|. \quad (1.8)$$

In Ref.[14], the inequalities (1.8) are strengthened for separable states: a necessary condition of separability, consisting of one-parameter family of inequalities, is thereby established for any two-mode state. Moreover, for the special class of TMGSs, the strongest of these inequalities is proven to be also a sufficient condition of separability. Some other necessary conditions of separability which employ pairs of more general non-local observables depending on more parameters have been pointed out [3, 15, 16]. They are expressed in terms of the product or the sum of the corresponding variances which include EPR-like correlations.

The aim of this paper is twofold. On the one hand, we tackle the characterization of the separability of TMGSs based on the analysis of EPR-like correlations. We formulate full criteria of separability for TMGSs, using both the product and the sum of variances of two EPR-like observables. On the other hand, we prove directly the equivalence between two approaches of the separability property of a TMGS: an EPR-like one, initiated by Duan *et al.* and developed in the present work, and the Simon PPT one. The plan of the paper is as follows. In Sec. II, we recall the EPR-like necessary conditions of separability for two-mode states, as discussed by Giovannetti *et al.* in Ref. [16]. Section III is an overview of several useful properties of the undisplaced TMGSs. In Sec. IV, the product function occurring in Eq. (1.3) and one of the sum functions from Eq. (1.8) are normalized according to the formulae presented in Sec. II. We evaluate here

their minimal values, which are manifest separability indicators for any TMGS owing to the PPT separability criterion [3]. A two-parameter regularized sum function is examined along the same lines in Sec. V, yielding another indicator of separability equally based on Simon's PPT criterion. The formulae established in Secs. IV and V allow us to derive the Peres-Simon PPT necessary condition of separability for a TMGS from each of the three EPR-like corresponding necessary conditions we have employed. In Sec. VI, we optimize a regularized two-mode sum-type correlation function within the framework of the EPR-like approach developed by Duan *et al.* in Ref. [14]. We find a necessary and sufficient condition of separability related to the scaled standard form II of the covariance matrix (CM). This EPR-like condition of separability is explicitly proven to be fully equivalent to Simon's PPT one. Section VII summarizes our main results insisting on the connection between the EPR-like and PPT separability conditions for a TMGS. Appendix A presents some nontrivial identities involving the standard-form parameters of the CM of a TMGS. In Appendix B, we prove the positivity of the Hessian matrices of the three EPR-like correlation functions examined in Secs. IV and V, when evaluated at their stationary points. Appendix C points out the existence of a scaled standard form II of the CM, which is exploited in Sec. VI.

II. SEPARABLE TWO-MODE STATES

Let us consider a pair of EPR-like observables which are linear combinations of the canonical quadrature operators of the two modes:

$$\begin{aligned}\hat{Q} &:= \alpha_1 \hat{q}_1 - \alpha_2 \hat{q}_2, & \hat{P} &:= \beta_1 \hat{p}_1 + \beta_2 \hat{p}_2, \\ (\alpha_j > 0, \beta_j > 0: j = 1, 2).\end{aligned}\quad (2.1)$$

The coordinates and momenta in Eq. (2.1) are defined in terms of the amplitude operators of the modes:

$$\begin{aligned}\hat{q}_j &:= \frac{1}{\sqrt{2}} (\hat{a}_j + \hat{a}_j^\dagger), & \hat{p}_j &:= \frac{1}{\sqrt{2}i} (\hat{a}_j - \hat{a}_j^\dagger), \\ (j = 1, 2).\end{aligned}\quad (2.2)$$

Obviously, Reid's pair of one-parameter observables built with independent parameters (1.1), as well as the single-parameter one (1.5) are particular cases of EPR-like operators (2.1). The commutation relation

$$[\hat{Q}, \hat{P}] = i(\alpha_1 \beta_1 - \alpha_2 \beta_2) \hat{I} \quad (2.3)$$

shows that the operators (2.1) are genuine EPR observables if and only if $\alpha_1 \beta_1 = \alpha_2 \beta_2$. The corresponding Heisenberg uncertainty relation,

$$\Delta Q \Delta P \geq \frac{1}{2} |\alpha_1 \beta_1 - \alpha_2 \beta_2|, \quad (2.4)$$

entails the sum-type inequality

$$(\Delta Q)^2 + (\Delta P)^2 \geq |\alpha_1 \beta_1 - \alpha_2 \beta_2|. \quad (2.5)$$

If the two-mode state is separable, i. e., it is a convex combination of product states,

$$\hat{\rho}_s := \sum_k w_k \hat{\rho}_1^{(k)} \otimes \hat{\rho}_2^{(k)}, \quad \left(w_k > 0, \quad \sum_k w_k = 1 \right),$$

then the product $(\Delta Q)_s (\Delta P)_s$ has a stronger lower bound than in Eq. (2.4) [16]:

$$(\Delta Q)_s (\Delta P)_s \geq \frac{1}{2} (\alpha_1 \beta_1 + \alpha_2 \beta_2). \quad (2.6)$$

Accordingly, for the sum of variances, an inequality stronger than Eq. (2.5) holds:

$$[(\Delta Q)_s]^2 + [(\Delta P)_s]^2 \geq \alpha_1 \beta_1 + \alpha_2 \beta_2. \quad (2.7)$$

It is useful to specialize the necessary conditions for the separability of a two-mode state, Eqs. (2.6) and (2.7), to the pairs of EPR-like observables (1.1) and (1.5). We get the following two sets of inequalities:

$$[\Delta Q(\lambda)]_s [\Delta P(\mu)]_s \geq \frac{1}{2} (1 + \lambda \mu), \quad (2.8)$$

$$\{[\Delta Q(\lambda)]_s\}^2 + \{[\Delta P(\mu)]_s\}^2 \geq 1 + \lambda \mu; \quad (2.9)$$

$$[\Delta Q(\alpha)]_s [\Delta P_{\pm}(\alpha)]_s \geq \frac{1}{2} \left(\alpha^2 + \frac{1}{\alpha^2} \right), \quad (2.10)$$

$$\{[\Delta Q(\alpha)]_s\}^2 + \{[\Delta P_{\pm}(\alpha)]_s\}^2 \geq \alpha^2 + \frac{1}{\alpha^2}. \quad (2.11)$$

The necessary conditions of separability (2.11) have first been derived in Ref. [14].

III. TWO-MODE GAUSSIAN STATES

For a long time, the Gaussian states (GSs) of the quantum radiation field are known to be of central importance in various areas of quantum optics. In general, the GSs play an important role for those quantum systems involving a quadratic bosonic Hamiltonian that generates correlations between bosonic modes. They are achieved in condensed matter, as well as in atomic ensembles such as trapped ions or Bose-Einstein condensates. The GSs of light are also largely employed in quantum information processing with continuous variables; their usefulness has been reviewed in Refs. [17–20].

Especially accessible and insightful are the TMGSs. Here we recollect just the strictly necessary notions and results [21–23] that enable us to discuss separability issues. Since these are not affected by translations in the

phase space, it is sufficient to deal with undisplaced (zero-mean) TMGSs.

Recall that the characteristic function (CF) of an undisplaced TMGS $\hat{\rho}$ is a real exponential:

$$\chi(u) = \exp \left[-\frac{1}{2} (Ju)^T \mathcal{V} (Ju) \right]. \quad (3.1)$$

Its argument is a dimensionless vector $u \in \mathbb{R}^4$ whose components are eigenvalues of the canonical quadrature operators of the modes:

$$u^T := (q_1, p_1, q_2, p_2). \quad (3.2)$$

J designates the standard matrix of the symplectic form on \mathbb{R}^4 , which is block-diagonal and skew-symmetric:

$$J := J_1 \oplus J_2 : \quad J_1 = J_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3.3)$$

The TMGS $\hat{\rho}$ is entirely specified, via the CF (3.1), by the real and symmetric 4×4 CM, which is denoted \mathcal{V} . Its entries are expectation values of products of the deviations from the means of the quadratures (2.2). The order of the rows and columns is indicated by the current vector (3.2). The CM \mathcal{V} fulfills the strong form (Robertson-Schrödinger) of the uncertainty relations for the canonical quadrature observables (2.2):

$$\mathcal{V} + \frac{i}{2} J \geq 0. \quad (3.4)$$

The above requirement that the complex matrix $\mathcal{V} + \frac{i}{2} J$ has to be positive semidefinite is a necessary and sufficient condition for the existence of the GS $\hat{\rho}$ [3]. It implies the inequality

$$\mathcal{D} := \det \left(\mathcal{V} + \frac{i}{2} J \right) \geq 0, \quad (3.5)$$

as well as the general property that the CM \mathcal{V} is positive definite: $\mathcal{V} > 0$. By contrast, the saturation equality $\mathcal{D} = 0$ is a quite special feature. However, it is shared by all the pure GSs and also by an interesting class of mixed ones. All these states are said to be at the physicality edge.

It is often convenient to partition the CM \mathcal{V} into 2×2 submatrices:

$$\mathcal{V} = \begin{pmatrix} \mathcal{V}_1 & \mathcal{C} \\ \mathcal{C}^T & \mathcal{V}_2 \end{pmatrix}. \quad (3.6)$$

Here \mathcal{V}_j , ($j = 1, 2$), denote the CMs of the individual single-mode reduced GSs, while \mathcal{C} displays the cross-correlations between the modes.

Let us mention that a symplectic transformation S of the canonical quadrature observables (2.2) in the Heisenberg picture induces a congruence transformation of any CM:

$$\mathcal{V}' = S \mathcal{V} S^T, \quad S \in \text{Sp}(4, \mathbb{R}). \quad (3.7)$$

At the same time, a unitary operator $\hat{U}(S)$ acting on the two-mode Fock space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is associated to the symplectic matrix S . Remarkably, the corresponding transformation of the TMGS in the Schrödinger picture, $\hat{\rho}' = \hat{U}(S) \hat{\rho} \hat{U}^\dagger(S)$, preserves the Gaussian nature of the state. Note that, if the symplectic transformation S consists of two separate single-mode ones,

$$S = S_1 \oplus S_2 : \quad S \in \text{Sp}(2, \mathbb{R}) \times \text{Sp}(2, \mathbb{R}), \quad (3.8)$$

then the above transformation of the TMGS $\hat{\rho}$ does not affect its amount of entanglement. It has been proven in Ref. [14] and just exploited in Ref. [3] the existence of a local symplectic transformation (3.8) that leads to a standard form of the CM:

$$\mathcal{V}(1, 1) := \begin{pmatrix} b_1 & 0 & c & 0 \\ 0 & b_1 & 0 & d \\ c & 0 & b_2 & 0 \\ 0 & d & 0 & b_2 \end{pmatrix}, \quad (3.9)$$

depending on four parameters b_1, b_2, c, d . With no loss of generality, one can choose $b_1 \geq b_2 > 0$ and $c \geq |d|$. Following Refs. [3, 14], we apply two independent one-mode squeeze transformations to the standard-form CM (3.9). Let us denote the corresponding scaling factors by

$$u_1 := e^{2r_1} \geq 1, \quad u_2 := e^{2r_2} \geq 1, \quad (3.10)$$

where r_1 and r_2 are the squeeze parameters in the two modes. The transformed state has a *scaled standard-form* CM with the block structure (3.6):

$$\mathcal{V}(u_1, u_2) = \begin{pmatrix} \mathcal{V}_1(u_1) & \mathcal{C}(\sqrt{u_1 u_2}) \\ \mathcal{C}(\sqrt{u_1 u_2}) & \mathcal{V}_2(u_2) \end{pmatrix}. \quad (3.11)$$

All its 2×2 submatrices are diagonal. The CMs of the single-mode reduced GSs read

$$\mathcal{V}_j(u_j) = \begin{pmatrix} b_j u_j & 0 \\ 0 & \frac{b_j}{u_j} \end{pmatrix}, \quad (j = 1, 2), \quad (3.12)$$

while the cross-correlation matrix is

$$\mathcal{C}(\sqrt{u_1 u_2}) = \begin{pmatrix} c\sqrt{u_1 u_2} & 0 \\ 0 & \frac{d}{\sqrt{u_1 u_2}} \end{pmatrix}. \quad (3.13)$$

The TMGSs whose CMs have a scaled standard form (3.11)–(3.13) constitute a class characterized by the set of four standard-form parameters $\{b_1, b_2, c, d\}$ of the given TMGS $\hat{\rho}$, which are $\text{Sp}(2, \mathbb{R}) \times \text{Sp}(2, \mathbb{R})$ -invariant. The states of this class are labeled by the pair of local squeeze factors $\{u_1, u_2\}$ [14]. They are locally unitary similar to the given TMGS $\hat{\rho}$ and thereby possess precisely its amount of entanglement.

The Robertson-Schrödinger uncertainty relation (3.4) reduces to the following restrictions for the standard-form parameters:

$$b_1 \geq \frac{1}{2}, \quad b_2 \geq \frac{1}{2}; \quad (3.14)$$

$$\begin{aligned} b_1 b_2 - c^2 &\geq \frac{1}{4} \max \left\{ \frac{b_1}{b_2}, \frac{b_2}{b_1} \right\} \geq \frac{1}{4}, \\ b_1 b_2 - d^2 &\geq \frac{1}{4} \max \left\{ \frac{b_1}{b_2}, \frac{b_2}{b_1} \right\} \geq \frac{1}{4}; \end{aligned} \quad (3.15)$$

$$\begin{aligned} \mathcal{D} &= (b_1 b_2 - c^2) (b_1 b_2 - d^2) \\ &\quad - \frac{1}{4} (b_1^2 + b_2^2 + 2cd) + \frac{1}{16} \geq 0. \end{aligned} \quad (3.16)$$

The inequalities (3.15) give rise to another one involving the symplectic invariant $\det(\mathcal{V}) = (b_1 b_2 - c^2) (b_1 b_2 - d^2)$ and thus concerning the purity of the state:

$$\det(\mathcal{V}) \geq \frac{1}{16} \iff \text{Tr}(\hat{\rho}^2) \leq 1. \quad (3.17)$$

According to Williamson's theorem [24], the CM \mathcal{V} of any TMGS $\hat{\rho}$ is congruent to a diagonal CM via a symplectic matrix (3.7), which is unique up to the sign. The corresponding diagonal entries, denoted κ_\pm , are positive, each one occurring twice. They are called the symplectic eigenvalues of the CM \mathcal{V} [25]. By virtue of Eq. (3.7), the symplectic invariants $\det(\mathcal{V})$ and \mathcal{D} factor as follows:

$$\det(\mathcal{V}) = (\kappa_+)^2 (\kappa_-)^2, \quad (3.18)$$

$$\mathcal{D} = \left[(\kappa_+)^2 - \frac{1}{4} \right] \left[(\kappa_-)^2 - \frac{1}{4} \right] \geq 0, \quad (3.19)$$

with

$$\kappa_+ \geq \kappa_- \geq \frac{1}{2}. \quad (3.20)$$

We employ Eqs. (3.18) and (3.19) to get the symplectic eigenvalues in terms of the standard-form parameters [25]:

$$\begin{aligned} (\kappa_\pm)^2 &= \frac{1}{2} \left[(b_1^2 + b_2^2 + 2cd) \pm \sqrt{\Delta} \right], \\ \Delta &:= (b_1^2 + b_2^2 + 2cd)^2 - 4 \det(\mathcal{V}) \\ &= (b_1^2 - b_2^2)^2 + 4(b_1 c + b_2 d)(b_2 c + b_1 d) \geq 0. \end{aligned} \quad (3.21)$$

In order to formulate Simon's separability criterion for TMGSs, we have to consider the partial transpose $\hat{\rho}^{\text{PT}}$ of the TMGS $\hat{\rho}$. Basically, partial transposition preserves the Gaussian character of the operator, does not modify the standard-form parameters b_1, b_2, c , and changes the sign of d . Therefore, $\hat{\rho}^{\text{PT}}$ is an undisplaced two-mode Gaussian operator whose CM \mathcal{V}^{PT} has the standard-form

parameters $\{b_1, b_2, c, -d\}$ [3]. First, Simon has proven a lemma asserting that any TMGS with $d \geq 0$ is separable. Then, he has shown that Peres' necessary condition of separability, which claims that $\hat{\rho}^{\text{PT}}$ should be a quantum state, i. e.,

$$\mathcal{V}^{\text{PT}} + \frac{i}{2}J \geq 0, \quad (3.22)$$

is also a *sufficient* one [3]. The condition (3.22) for the existence of a GS $\hat{\rho}^{\text{PT}}$ reduces to the inequality

$$\mathcal{D}^{\text{PT}} := \det \left(\mathcal{V}^{\text{PT}} + \frac{i}{2}J \right) \geq 0, \quad (3.23)$$

which is Simon's criterion of separability. It is usually written in the form (3.16) with $d \rightarrow -d$. Therefore, a TMGS is separable if and only if the following inequality is fulfilled:

$$\mathcal{D}^{\text{PT}} = \det(\mathcal{V}) - \frac{1}{4}(b_1^2 + b_2^2 - 2cd) + \frac{1}{16} \geq 0. \quad (3.24)$$

Accordingly, the identity

$$\mathcal{D}^{\text{PT}} = \mathcal{D} + cd \quad (3.25)$$

displays the property that all the TMGSs with $d \geq 0$ are separable, in agreement with Simon's lemma [3].

IV. NORMALIZED SEPARABILITY INDICATORS

We write the variances of the EPR-like observables (2.1) for a TMGS whose CM has a scaled standard form (3.11)– (3.13):

$$(\Delta Q)^2 = \alpha_1^2 b_1 u_1 + \alpha_2^2 b_2 u_2 - 2\alpha_1 \alpha_2 c \sqrt{u_1 u_2}, \quad (4.1)$$

$$(\Delta P)^2 = \beta_1^2 \frac{b_1}{u_1} + \beta_2^2 \frac{b_2}{u_2} + 2\beta_1 \beta_2 \frac{d}{\sqrt{u_1 u_2}}. \quad (4.2)$$

Remark that, while all the TMGSs (3.11)– (3.13) belonging to a class with fixed standard-form parameters possess the same amount of entanglement, their EPR-like variances (4.1) and (4.2) depend, in addition, on the local squeezings u_1, u_2 . In what follows, we consider two functions which are normalized as suggested by the separability lower bounds (2.6) and (2.7):

$$E(\alpha_1 \beta_1, \alpha_2 \beta_2, u_1, u_2) := \frac{(\Delta Q)^2 (\Delta P)^2}{(\alpha_1 \beta_1 + \alpha_2 \beta_2)^2}, \quad (4.3)$$

$$F(\alpha_1 \beta_1, \alpha_2 \beta_2, u_1, u_2) := \frac{(\Delta Q)^2 + (\Delta P)^2}{\alpha_1 \beta_1 + \alpha_2 \beta_2}. \quad (4.4)$$

According to Eqs. (2.6) and (2.7), for any separable two-mode state (Gaussian or non-Gaussian), the following inequalities are satisfied:

$$E \geq \frac{1}{4}, \quad F \geq 1. \quad (4.5)$$

In the sequel, we develop a two-step program. First, in order to handle simpler functions, we diminish the number of their independent variables as much as possible by making suitable substitutions or choices. Second, we find the absolute minima of the resulting simpler functions, which prove to be manifest separability indicators. It is sufficient to restrict our search in this step to values $d < 0$ in the variance (4.2).

A. Product form

We start by absorbing the scaling factors in Eqs. (4.1) and (4.2) into four new positive parameters,

$$\alpha'_j = \alpha_j \sqrt{u_j}, \quad \beta'_j = \beta_j \frac{1}{\sqrt{u_j}}, \quad (j = 1, 2), \quad (4.6)$$

so that the variances become:

$$(\Delta Q)^2 = b_1 (\alpha'_1)^2 + b_2 (\alpha'_2)^2 - 2c \alpha'_1 \alpha'_2, \quad (4.7)$$

$$(\Delta P)^2 = b_1 (\beta'_1)^2 + b_2 (\beta'_2)^2 + 2d \beta'_1 \beta'_2. \quad (4.8)$$

Note the identity

$$\alpha_1 \beta_1 + \alpha_2 \beta_2 = \alpha'_1 \beta'_1 + \alpha'_2 \beta'_2. \quad (4.9)$$

After replacing Eqs. (4.7)– (4.9) into Eq. (4.3), the function $E(\alpha'_1 \beta'_1, \alpha'_2 \beta'_2)$ can be further simplified. As a matter of fact, it is a function of two positive variables,

$$\lambda := \frac{\alpha'_2}{\alpha'_1}, \quad \mu := \frac{\beta'_2}{\beta'_1}, \quad (4.10)$$

which reads:

$$E(\lambda, \mu) := \frac{[\Delta Q(\lambda)]^2 [\Delta P(\mu)]^2}{(1 + \lambda \mu)^2}. \quad (4.11)$$

The numerator of the fraction in the r. h. s. of Eq. (4.11) is equal to the product of the variances of Reid's EPR-like observables (1.1) in a standard-form TMGS (3.9):

$$[\Delta Q(\lambda)]^2 = b_1 + b_2 \lambda^2 - 2c \lambda, \quad (4.12)$$

$$[\Delta P(\mu)]^2 = b_1 + b_2 \mu^2 + 2d \mu. \quad (4.13)$$

We prove the following statement.

Theorem 1. *A TMGS is separable if the absolute minimum of the function $E(\lambda, \mu)$ is greater than or at least equal to $1/4$:*

$$E_m := \min_{\{\lambda, \mu\}} E(\lambda, \mu) \geq \frac{1}{4}. \quad (4.14)$$

Proof. The first-order derivatives

$$\begin{aligned}\frac{\partial \ln(E)}{\partial \lambda} &= \frac{2(b_2\lambda - c)}{b_1 + b_2\lambda^2 - 2c\lambda} - \frac{2\mu}{1 + \lambda\mu}, \\ \frac{\partial \ln(E)}{\partial \mu} &= \frac{2(b_2\mu + d)}{b_1 + b_2\mu^2 + 2d\mu} - \frac{2\lambda}{1 + \lambda\mu}\end{aligned}\quad (4.15)$$

vanish at a stationary point of the function (4.11). The resulting equations,

$$\begin{aligned}[\Delta Q(\lambda)]^2 &= \frac{1}{\mu} (b_2\lambda - c) (1 + \lambda\mu), \\ [\Delta P(\mu)]^2 &= \frac{1}{\lambda} (b_2\mu + d) (1 + \lambda\mu),\end{aligned}\quad (4.16)$$

have a unique solution:

$$\lambda_m = \frac{(b_1^2 - b_2^2) + \sqrt{\Delta^{\text{PT}}}}{2(b_1c - b_2d)}, \quad \mu_m = \frac{(b_1^2 - b_2^2) + \sqrt{\Delta^{\text{PT}}}}{2(b_2c - b_1d)}, \quad (4.17)$$

where Δ^{PT} is the discriminant in Eq. (A5). In Appendix B we have proven that the value (B4),

$$E_m = \frac{(b_2\lambda_m - c)(b_2\mu_m + d)}{\lambda_m\mu_m}, \quad (4.18)$$

is the absolute minimum of the function $E(\lambda, \mu)$. By use of Eqs. (4.17) and (A5), we find the formula

$$E_m = (\kappa_-^{\text{PT}})^2. \quad (4.19)$$

When $d < 0$, Simon's separability condition (3.24) in conjunction with Eq. (4.19) gives the alternative:

$$E_m \geq \frac{1}{4} \iff \hat{\rho} \text{ separable}, \quad (d < 0),$$

$$E_m < \frac{1}{4} \iff \hat{\rho} \text{ entangled}, \quad (d < 0).$$

As $(\kappa_-^{\text{PT}})^2 \geq \frac{1}{4} \iff \mathcal{D}^{\text{PT}} \geq 0$, the first inequality holds also for $d \geq 0$ due to the identity (3.25). This result is in accordance with Simon's lemma regarding the separability of any TMGS with $d \geq 0$ [3]. We put together the above results to draw the general conclusion:

$$\begin{aligned}E_m \geq \frac{1}{4} &\iff \hat{\rho} \text{ separable}, \\ E_m < \frac{1}{4} &\iff \hat{\rho} \text{ entangled}.\end{aligned}\quad (4.20)$$

This necessary and sufficient condition of separability for TMGSs concludes the proof.

Therefore, the minimum of the normalized product (4.11) of two EPR-like uncertainties is equal to the square of the smallest symplectic eigenvalue κ_-^{PT} . Owing to the PPT separability criterion, it is itself a separability indicator. A previous relationship between κ_-^{PT} and a product of EPR-like uncertainties was found by a different

treatment and from another perspective in Ref. [26]. Quite recently, the present approach was successfully applied in Ref. [27] to a special class of TMGSs, namely, the squeezed thermal states (STSS): $c = -d > 0$ [22].

An interesting related problem for TMGSs regarding steerability criteria was recently put forward by Kogias and Adesso [28]. These authors have considered the product of EPR-like uncertainties occurring in the Reid condition (1.4) and have minimized it with respect to the pair of parameters $\{\lambda, \mu\}$ and the local variables of a TMGS. They have thus recovered the explicit symplectically invariant formula for the condition of steerability of GSs by Gaussian measurements which has first been written in Refs. [11, 12].

B. Sum form

We find it convenient to simplify the function (4.4) by setting $\alpha_1 = \alpha_2 := \alpha$, $\beta_1 = \beta_2 = \frac{1}{\alpha}$:

$$F(\alpha^2, u_1, u_2) := \frac{[\Delta Q(\alpha)]^2 + [\Delta P_+(\alpha)]^2}{\alpha^2 + \frac{1}{\alpha^2}}. \quad (4.21)$$

The correlation function (4.21) is built with the variances of a pair of non-local one-parameter observables (1.5) introduced in Ref. [14]:

$$[\Delta Q(\alpha)]^2 = b_1 u_1 \alpha^2 + b_2 u_2 \frac{1}{\alpha^2} - 2c\sqrt{u_1 u_2}, \quad (4.22)$$

$$[\Delta P_{\pm}(\alpha)]^2 = \frac{b_1}{u_1} \alpha^2 + \frac{b_2}{u_2} \frac{1}{\alpha^2} \pm \frac{2d}{\sqrt{u_1 u_2}}. \quad (4.23)$$

Therefore, it reads:

$$F(\alpha^2, u_1, u_2) = \frac{1}{\alpha^4 + 1} \left[b_1 \left(u_1 + \frac{1}{u_1} \right) \alpha^4 - 2(c\sqrt{u_1 u_2} - d \frac{1}{\sqrt{u_1 u_2}}) \alpha^2 + b_2 \left(u_2 + \frac{1}{u_2} \right) \right]. \quad (4.24)$$

We state

Theorem 2. *A TMGS is separable if the absolute minimum of the function $F(\alpha^2, u_1, u_2)$ is greater than or at least equal to 1:*

$$F_m := \min_{\{\alpha^2, u_1, u_2\}} F(\alpha^2, u_1, u_2) \geq 1. \quad (4.25)$$

Proof. The first-order derivatives

$$\begin{aligned}\frac{\partial F}{\partial (\alpha^2)} &= \frac{2}{(\alpha^4 + 1)^2} \left\{ \left(c\sqrt{u_1 u_2} - \frac{d}{\sqrt{u_1 u_2}} \right) (\alpha^4 - 1) \right. \\ &\quad \left. + \left[b_1 \left(u_1 + \frac{1}{u_1} \right) - b_2 \left(u_2 + \frac{1}{u_2} \right) \right] \alpha^2 \right\},\end{aligned}\quad (4.26)$$

$$\begin{aligned}\frac{\partial F}{\partial u_1} &= \frac{\alpha^2}{(\alpha^4 + 1) u_1} \left[b_1 \left(u_1 - \frac{1}{u_1} \right) \alpha^2 \right. \\ &\quad \left. - \left(c\sqrt{u_1 u_2} + \frac{d}{\sqrt{u_1 u_2}} \right) \right],\end{aligned}\quad (4.27)$$

$$\frac{\partial F}{\partial u_2} = \frac{\alpha^2}{(\alpha^4 + 1) u_2} \left[b_2 \left(u_2 - \frac{1}{u_2} \right) \frac{1}{\alpha^2} - \left(c\sqrt{u_1 u_2} + \frac{d}{\sqrt{u_1 u_2}} \right) \right] \quad (4.28)$$

vanish at a stationary point of the function (4.24). We try to solve the resulting system of stationarity equations:

$$\left(c\sqrt{u_1 u_2} - \frac{d}{\sqrt{u_1 u_2}} \right) (1 - \alpha^4) = \left[b_1 \left(u_1 + \frac{1}{u_1} \right) - b_2 \left(u_2 + \frac{1}{u_2} \right) \right] \alpha^2, \quad (4.29)$$

$$b_1 \left(u_1 - \frac{1}{u_1} \right) \alpha^2 = c\sqrt{u_1 u_2} + \frac{d}{\sqrt{u_1 u_2}}, \quad (4.30)$$

$$b_2 \left(u_2 - \frac{1}{u_2} \right) \frac{1}{\alpha^2} = c\sqrt{u_1 u_2} + \frac{d}{\sqrt{u_1 u_2}}. \quad (4.31)$$

From Eqs. (4.30) and (4.31) it follows:

$$\alpha^4 = \frac{b_2 \left(u_2 - \frac{1}{u_2} \right)}{b_1 \left(u_1 - \frac{1}{u_1} \right)}; \quad (4.32)$$

$$b_1 b_2 \left(u_1 - \frac{1}{u_1} \right) \left(u_2 - \frac{1}{u_2} \right) = \left(c\sqrt{u_1 u_2} + \frac{d}{\sqrt{u_1 u_2}} \right)^2; \quad (4.33)$$

$$[\Delta P_+(\alpha)]^2 = [\Delta Q(\alpha)]^2. \quad (4.34)$$

Insertion of Eqs. (4.30) and (4.32) into Eq. (4.29) gives the proportionality relation

$$u_2 = \gamma u_1, \quad \gamma := \frac{b_2 c - b_1 d}{b_1 c - b_2 d} \leq 1. \quad (4.35)$$

By substituting it into Eq. (4.33), we get a quadratic equation in the product $p := u_1 u_2 \geq 1$:

$$(b_1 b_2 - c^2) p^2 - \left[b_1 b_2 \left(\gamma + \frac{1}{\gamma} \right) + 2cd \right] p + (b_1 b_2 - d^2) = 0 \quad (4.36)$$

with

$$\gamma + \frac{1}{\gamma} = 2 + \frac{[(b_1 - b_2)(c + d)]^2}{(b_1 c - b_2 d)(b_2 c - b_1 d)}. \quad (4.37)$$

Let us denote by Δ_p the discriminant of the quadratic trinomial in Eq. (4.36) and let p_{\pm} be its roots. Making use of Eq. (A5), we find the relation

$$\Delta_p = \left[\frac{b_1 b_2 (c^2 - d^2)}{(b_1 c - b_2 d)(b_2 c - b_1 d)} \right]^2 \Delta^{\text{PT}} \geq 0. \quad (4.38)$$

Since $p_- < 1$ for $c + d > 0$, the only acceptable solution of the quadratic equation (4.36) is

$$p_+ = \frac{c(b_1 b_2 - d^2) - d(\kappa_-^{\text{PT}})^2}{-d(b_1 b_2 - c^2) + c(\kappa_-^{\text{PT}})^2} \geq 1. \quad (4.39)$$

With Eqs. (4.35) and (4.39) we get the scaling factors:

$$u_{1m} = \left[\frac{b_2(b_1 b_2 - d^2) - b_1(\kappa_-^{\text{PT}})^2}{b_2(b_1 b_2 - c^2) - b_1(\kappa_-^{\text{PT}})^2} \right]^{\frac{1}{2}},$$

$$u_{2m} = \left[\frac{b_1(b_1 b_2 - d^2) - b_2(\kappa_-^{\text{PT}})^2}{b_1(b_1 b_2 - c^2) - b_2(\kappa_-^{\text{PT}})^2} \right]^{\frac{1}{2}}. \quad (4.40)$$

In view of Eq. (A5), they have the alternative expressions:

$$u_{1m} = \left\{ \frac{b_1 [\sqrt{\Delta^{\text{PT}}} - (b_1^2 - b_2^2)] + 2d(b_1 c - b_2 d)}{b_1 [\sqrt{\Delta^{\text{PT}}} - (b_1^2 - b_2^2)] - 2c(b_2 c - b_1 d)} \right\}^{\frac{1}{2}},$$

$$u_{2m} = \left\{ \frac{b_2 [\sqrt{\Delta^{\text{PT}}} + (b_1^2 - b_2^2)] + 2d(b_2 c - b_1 d)}{b_2 [\sqrt{\Delta^{\text{PT}}} + (b_1^2 - b_2^2)] - 2c(b_1 c - b_2 d)} \right\}^{\frac{1}{2}}. \quad (4.41)$$

Insertion of the scaling factors (4.41) into Eq. (4.32) leads to the following value of the EPR-like parameter α :

$$\alpha_m = \left[\frac{\sqrt{\Delta^{\text{PT}}} - (b_1^2 - b_2^2)}{\sqrt{\Delta^{\text{PT}}} + (b_1^2 - b_2^2)} \right]^{\frac{1}{4}} \leq 1. \quad (4.42)$$

There is therefore a single stationary value of the function $F(\alpha^2, u_1, u_2)$, Eq. (4.24):

$$F_m = F(\alpha_m^2, u_{1m}, u_{2m}). \quad (4.43)$$

In fact, this is its absolute minimum, as proven in Appendix B. With the solution (4.40) and (4.42), we take advantage of Eqs. (4.34), (A6), and (A8) to find the minimum value

$$F_m = 2\kappa_-^{\text{PT}}. \quad (4.44)$$

By the same argument as before Eq. (4.20), the PPT separability condition (3.24) implies, via Eq. (4.44), the alternative:

$$F_m \geq 1 \iff \hat{\rho} \text{ separable},$$

$$F_m < 1 \iff \hat{\rho} \text{ entangled}. \quad (4.45)$$

This concludes the proof.

It is worth mentioning that Eqs. (4.41) and (4.42), which specify the coordinates of the minimum point, become much simpler for two classes of TMGSs, namely, the STSs and the symmetric states. We find it convenient to write down here the corresponding formulae.

1. *Two-mode STSs:*

$$\begin{aligned} c = -d > 0 &\iff u_{1m} = u_{2m} = 1 : \quad \gamma = 1; \\ \sqrt{\Delta^{\text{PT}}} &= (b_1 + b_2) \sqrt{\delta}, \quad \delta := (b_1 - b_2)^2 + 4c^2 > 0 : \\ (\alpha_m)^2 &= \frac{1}{2c} \left[\sqrt{\delta} - (b_1 - b_2) \right]. \end{aligned} \quad (4.46)$$

2. *Symmetric TMGSs:*

$$\begin{aligned} b_1 = b_2 =: b &\implies u_{1m} = u_{2m} = \sqrt{\frac{b+d}{b-c}} : \quad \gamma = 1; \\ \sqrt{\Delta^{\text{PT}}} &= 2b(c-d) > 0, \quad \alpha_m = 1. \end{aligned} \quad (4.47)$$

It is interesting to notice that in Ref.[23] the minimal EPR-like uncertainty in sum form was explicitly evaluated for symmetric TMGSs. The authors have interpreted therein the smallest symplectic eigenvalue of the partially transposed state as a quantifier of the greatest amount of EPR correlations which can be created in a symmetric TMGS by means of local operations. Our present findings in the general case, Eq. (4.19) for the product form of the normalized EPR-like uncertainties, and Eq. (4.44) for the sum-form ones, appear to be consistent with this interpretation.

To complete this section, we write the appropriate necessary conditions of separability (4.5) for a TMGS:

$$E(\lambda, \mu) \geq \frac{1}{4}, \quad F(\alpha^2, u_1, u_2) \geq 1. \quad (4.48)$$

When $d < 0$, these inequalities hold for any values of the above functions, including their minima (4.19) and (4.44), respectively:

$$E_m = (\kappa_-^{\text{PT}})^2 \geq \frac{1}{4}, \quad F_m = 2\kappa_-^{\text{PT}} \geq 1. \quad (4.49)$$

Each inequality (4.49) is equivalent to the Peres-Simon necessary condition of separability:

$$\kappa_-^{\text{PT}} \geq \frac{1}{2} \iff \mathcal{D}^{\text{PT}} \geq 0. \quad (4.50)$$

When $d \geq 0$, Eq. (4.50) is equally valid, owing to the identity (3.25). To sum up, for any TMGS, both the EPR-like necessary conditions of separability (4.48) imply the PPT condition (4.50).

On the other hand, Theorems 1 and 2 yield the above absolute minima (4.49). Accordingly, Simon's separability criterion for a TMGS (3.24) implies that each identity (4.48) is itself a sufficient condition of separability, as implicitly expressed by Eqs. (4.20) and (4.45).

V. SEPARABILITY INDICATOR IN REGULARIZED SUM FORM

We start from the Reid's pair of EPR-like observables (1.1). One gets their variances by setting $\alpha_1 = 1$, $\alpha_2 =: \lambda$, $\beta_1 = 1$, $\beta_2 =: \mu$ in Eqs. (4.1) and (4.2):

$$(\Delta Q)^2 = b_1 u_1 + b_2 u_2 \lambda^2 - 2c\sqrt{u_1 u_2} \lambda, \quad (5.1)$$

$$(\Delta P)^2 = \frac{b_1}{u_1} + \frac{b_2}{u_2} \mu^2 + 2\frac{d}{\sqrt{u_1 u_2}} \mu. \quad (5.2)$$

We assume that $d < 0$ in Eq. (5.2) and in the subsequent ones. It is convenient to absorb the scaling factor u_2 into a pair of new EPR-like parameters:

$$\xi := \sqrt{u_2} \lambda > 0, \quad \eta := \frac{1}{\sqrt{u_2}} \mu > 0. \quad (5.3)$$

This simplifies the variances (5.1) and (5.2) as follows:

$$[\Delta Q(\xi)]^2 = b_1 u_1 + b_2 \xi^2 - 2c\sqrt{u_1} \xi, \quad (5.4)$$

$$[\Delta P(\eta)]^2 = \frac{b_1}{u_1} + b_2 \eta^2 + 2\frac{d}{\sqrt{u_1}} \eta. \quad (5.5)$$

We take inspiration from the separability condition (2.7) to build an EPR-like correlation function of the remaining independent variables ξ, η, u_1 :

$$G(\xi, \eta, u_1) := [\Delta Q(\xi)]^2 + [\Delta P(\eta)]^2 - (1 + \xi\eta). \quad (5.6)$$

The function (5.6), which is non-negative for any separable two-mode state, has the explicit form:

$$\begin{aligned} G(\xi, \eta, u_1) &= (b_1 u_1 + b_2 \xi^2 - 2c\sqrt{u_1} \xi) \\ &+ \left(\frac{b_1}{u_1} + b_2 \eta^2 + 2\frac{d}{\sqrt{u_1}} \eta \right) - (1 + \xi\eta). \end{aligned} \quad (5.7)$$

The following statement is true.

Theorem 3. *A TMGS is separable if the absolute minimum of the function $G(\xi, \eta, u_1)$ is non-negative:*

$$G_m := \min_{\{\xi, \eta, u_1\}} G(\xi, \eta, u_1) \geq 0. \quad (5.8)$$

Proof. We write the stationarity conditions:

$$\begin{aligned} \frac{\partial G}{\partial \xi} = 0 : \quad & 2b_2 \xi - \eta = 2c\sqrt{u_1}, \\ \frac{\partial G}{\partial \eta} = 0 : \quad & -\xi + 2b_2 \eta = -2\frac{d}{\sqrt{u_1}}, \\ \frac{\partial G}{\partial u_1} = 0 : \quad & cu_1 \xi + d\eta = \frac{b_1}{\sqrt{u_1}} (u_1^2 - 1). \end{aligned} \quad (5.9)$$

The system (5.9) is linear in the variables (5.3). By solving the first two equations with respect to ξ and η and then replacing the result into the third one, we find its unique solution:

$$\begin{aligned} \xi_m &= \frac{1}{b_2^2 - \frac{1}{4}} \left(b_2 c \sqrt{u_{1m}} - \frac{1}{2} \frac{d}{\sqrt{u_{1m}}} \right), \\ \eta_m &= \frac{1}{b_2^2 - \frac{1}{4}} \left(\frac{1}{2} c \sqrt{u_{1m}} - b_2 \frac{d}{\sqrt{u_{1m}}} \right), \\ u_{1m} &= \left[\frac{b_2 (b_1 b_2 - d^2) - \frac{1}{4} b_1}{b_2 (b_1 b_2 - c^2) - \frac{1}{4} b_1} \right]^{\frac{1}{2}}. \end{aligned} \quad (5.10)$$

In Appendix B we have proven that the only stationary value of the function (5.7),

$$G_m = G(\xi_m, \eta_m, u_{1m}), \quad (5.11)$$

is its absolute minimum. Insertion of the coordinates (5.10) into Eq. (5.7) and a subsequent straightforward calculation yields the formula

$$G_m = \frac{1}{b_2^2 - \frac{1}{4}} \left(2 \left\{ \left[b_2 (b_1 b_2 - c^2) - \frac{1}{4} b_1 \right] \times \left[b_2 (b_1 b_2 - d^2) - \frac{1}{4} b_1 \right] \right\}^{\frac{1}{2}} - \left(b_2^2 - \frac{1}{4} - cd \right) \right). \quad (5.12)$$

Making use of the identity (A10), the minimum (5.12) can be cast into the form:

$$G_m = 4 \mathcal{D}^{\text{PT}} \left(2 \left\{ \left[b_2 (b_1 b_2 - c^2) - \frac{1}{4} b_1 \right] \times \left[b_2 (b_1 b_2 - d^2) - \frac{1}{4} b_1 \right] \right\}^{\frac{1}{2}} + \left(b_2^2 - \frac{1}{4} - cd \right) \right)^{-1}. \quad (5.13)$$

Because $d < 0$, Eq. (5.13) displays the feature that the minimum value G_m and the symplectic invariant \mathcal{D}^{PT} have the same sign. On the other hand, when $d \geq 0$, the TMGS is separable owing to Simon's lemma. Then, on account of Eqs. (2.7) and (3.25), both quantities are non-negative. Accordingly, the PPT condition of separability (3.24) imposes the alternative:

$$\begin{aligned} G_m \geq 0 &\iff \hat{\rho} \text{ separable,} \\ G_m < 0 &\iff \hat{\rho} \text{ entangled.} \end{aligned} \quad (5.14)$$

The proof is complete.

We conclude this section by writing the above-mentioned condition of separability: if a given TMGS $\hat{\rho}$ is separable, then the function (5.7) is non-negative,

$$G(\xi, \eta, u_1) \geq 0. \quad (5.15)$$

It follows the alternative: if $d < 0$, then its minimum (5.13) is also non-negative; if $d \geq 0$, then, owing to Eq. (3.25), the determinant \mathcal{D}^{PT} is non-negative:

$$\begin{aligned} d < 0 : \quad G_m \geq 0 &\iff \mathcal{D}^{\text{PT}} \geq 0; \\ d \geq 0 : \quad \mathcal{D}^{\text{PT}} = \mathcal{D} + cd &\geq 0. \end{aligned} \quad (5.16)$$

Therefore, the EPR-like necessary condition of separability (5.15), valid for any TMGS, implies the Peres-Simon PPT condition (4.50).

On the other hand, the absolute minimum (5.13) given by Theorem 3, shows that Simon's separability criterion for a TMGS, Eq. (4.50), entails that the identity (5.15) is itself a sufficient condition of separability, as implicitly expressed by Eq. (5.14).

VI. CONDITIONS OF SEPARABILITY: EPR-LIKE VERSUS PPT

We find it appropriate to develop the line of reasoning put forward by Duan *et al.* in their important work, Ref. [14]. We adopt their definitions and notations and employ Eq. (1.5) to introduce the non-local operator

$$\hat{P}(\alpha) := \begin{cases} \hat{P}_+(\alpha), & (d < 0), \\ \hat{P}_-(\alpha), & (d \geq 0). \end{cases} \quad (6.1)$$

Its variance,

$$[\Delta P(\alpha)]^2 = \frac{b_1}{u_1} \alpha^2 + \frac{b_2}{u_2} \frac{1}{\alpha^2} - \frac{2|d|}{\sqrt{u_1 u_2}}, \quad (6.2)$$

is the minimum over the pair of variances (4.23) with respect to the sign of d :

$$\Delta P(\alpha) = \min_{\{\text{sgn}(d)\}} \{ \Delta P_+(\alpha), \Delta P_-(\alpha) \}. \quad (6.3)$$

Recall the definition of the signum function of a real variable:

$$\text{sgn}(x) := \begin{cases} -1, & (x < 0), \\ 0, & (x = 0), \\ 1, & (x > 0). \end{cases}$$

Guided by the necessary conditions of separability (2.11) derived by Duan *et al.* in Ref. [14], we employ the following EPR-like correlation function, which is a regularized sum:

$$\begin{aligned} K(\alpha^2, u_1, u_2) &:= [\Delta Q(\alpha)]^2 + [\Delta P(\alpha)]^2 \\ &- \left(\alpha^2 + \frac{1}{\alpha^2} \right). \end{aligned} \quad (6.4)$$

Here the variances $[\Delta Q(\alpha)]^2$ and $[\Delta P(\alpha)]^2$ have the expressions (4.22) and (6.2), respectively. We apply the same pattern as in Sec. V, starting from the necessary condition of separability (2.11) which reads:

$$K(\alpha^2, u_1, u_2) \geq 0. \quad (6.5)$$

Substitution of the variances (4.22) and (6.2) into Eq. (6.4) gives the explicit formula:

$$\begin{aligned} K(\alpha^2, u_1, u_2) &= \alpha^2 \left[b_1 \left(u_1 + \frac{1}{u_1} \right) - 1 \right] \\ &+ \frac{1}{\alpha^2} \left[b_2 \left(u_2 + \frac{1}{u_2} \right) - 1 \right] \\ &- 2 \left(c \sqrt{u_1 u_2} + \frac{|d|}{\sqrt{u_1 u_2}} \right). \end{aligned} \quad (6.6)$$

Our approach is to minimize the function (6.6). This reaches its minimum with respect to the variable α^2 for the value

$$\tilde{\alpha}_m^2 = \sqrt{\frac{b_2 \left(u_2 + \frac{1}{u_2} \right) - 1}{b_1 \left(u_1 + \frac{1}{u_1} \right) - 1}}. \quad (6.7)$$

The obtained minimum is a function of the scaling factors:

$$\begin{aligned} f(u_1, u_2) &:= K(\tilde{\alpha}_m^2, u_1, u_2) \\ &= 2 \left\{ \left[b_1 \left(u_1 + \frac{1}{u_1} \right) - 1 \right] \left[b_2 \left(u_2 + \frac{1}{u_2} \right) - 1 \right] \right\}^{\frac{1}{2}} \\ &\quad - 2 \left(c\sqrt{u_1 u_2} + \frac{|d|}{\sqrt{u_1 u_2}} \right). \end{aligned} \quad (6.8)$$

The stationarity conditions for the function (6.8) reduce to the following system of equations in the unknowns u_1 and u_2 :

$$\frac{\frac{b_1}{u_1} - \frac{1}{2}}{b_1 u_1 - \frac{1}{2}} = \frac{\frac{b_2}{u_2} - \frac{1}{2}}{b_2 u_2 - \frac{1}{2}}, \quad (6.9)$$

$$\begin{aligned} &\sqrt{\left(b_1 u_1 - \frac{1}{2} \right) \left(b_2 u_2 - \frac{1}{2} \right)} - \sqrt{\left(\frac{b_1}{u_1} - \frac{1}{2} \right) \left(\frac{b_2}{u_2} - \frac{1}{2} \right)} \\ &= c\sqrt{u_1 u_2} - \frac{|d|}{\sqrt{u_1 u_2}}. \end{aligned} \quad (6.10)$$

Making use of Eq. (6.9), it is convenient to replace Eq. (6.10) by a polynomial one:

$$b_1 b_2 (u_1^2 - 1)(u_2^2 - 1) = (cu_1 u_2 - |d|)^2. \quad (6.11)$$

Equations (6.9) and (6.10) coincide with those written by Duan *et al.* in Ref. [14] in order to define what they have called the standard form II of the CM of a TMGS. We state here a theorem that sharpens a result of Ref. [14] regarding the existence of a solution of this system in the general case.

Theorem 4. *For any TMGS there exists at least a solution $\{\tilde{u}_1, \tilde{u}_2\}$ of the algebraic system (6.9) and (6.11) in the classicality range of the local squeeze factors:*

$$\tilde{u}_1 \in [1, 2b_1], \quad \tilde{u}_2 \in [1, 2b_2].$$

The proof of this important theorem is given in Appendix C. However, when trying to solve analytically the system under discussion in the general case, one faces a non-trivial algebraic equation of degree eight.

The standard form II of the CM is important because the condition of classicality for the corresponding TMGS,

$$\mathcal{V}(\tilde{u}_1, \tilde{u}_2) - \frac{1}{2}I_4 \geq 0, \quad (6.12)$$

is equivalent to the requirement

$$\tilde{f} := f(\tilde{u}_1, \tilde{u}_2) \geq 0. \quad (6.13)$$

Indeed, taking account of Eqs. (6.9) and (6.11), we get the twin formulae:

$$\begin{aligned} \tilde{f} &= 4 \left[\sqrt{\left(b_1 \tilde{u}_1 - \frac{1}{2} \right) \left(b_2 \tilde{u}_2 - \frac{1}{2} \right)} - c\sqrt{\tilde{u}_1 \tilde{u}_2} \right], \\ \tilde{f} &= 4 \left[\sqrt{\left(\frac{b_1}{\tilde{u}_1} - \frac{1}{2} \right) \left(\frac{b_2}{\tilde{u}_2} - \frac{1}{2} \right)} - \frac{|d|}{\sqrt{\tilde{u}_1 \tilde{u}_2}} \right]. \end{aligned} \quad (6.14)$$

We find it suitable to introduce the parallel notations:

$$\begin{aligned} \tilde{f}' &:= 4 \left[\sqrt{\left(b_1 \tilde{u}_1 - \frac{1}{2} \right) \left(b_2 \tilde{u}_2 - \frac{1}{2} \right)} + c\sqrt{\tilde{u}_1 \tilde{u}_2} \right] > 0, \\ \tilde{f}'' &:= 4 \left[\sqrt{\left(\frac{b_1}{\tilde{u}_1} - \frac{1}{2} \right) \left(\frac{b_2}{\tilde{u}_2} - \frac{1}{2} \right)} + \frac{|d|}{\sqrt{\tilde{u}_1 \tilde{u}_2}} \right] \geq 0. \end{aligned} \quad (6.15)$$

The matrix condition (6.12) reduces to four inequalities:

$$\begin{aligned} &b_1 \tilde{u}_1 - \frac{1}{2} \geq 0, \\ &\left(b_1 \tilde{u}_1 - \frac{1}{2} \right) \left(\frac{b_1}{\tilde{u}_1} - \frac{1}{2} \right) \geq 0, \\ &\left(\frac{b_1}{\tilde{u}_1} - \frac{1}{2} \right) 4^{-2} (\tilde{f} \tilde{f}') \geq 0, \\ &\det \left[\mathcal{V}(\tilde{u}_1, \tilde{u}_2) - \frac{1}{2}I_4 \right] = 4^{-4} (\tilde{f} \tilde{f}') (\tilde{f} \tilde{f}'') \geq 0. \end{aligned} \quad (6.16)$$

Three of them are already satisfied, so that the only condition to be fulfilled remains $\tilde{f} \geq 0$, Eq. (6.13). This classicality requirement is a sufficient condition for the separability of the given TMGS $\hat{\rho}$, whose CM is congruent with its standard form II $\mathcal{V}(\tilde{u}_1, \tilde{u}_2)$ via a local symplectic transformation (3.8). Moreover, by virtue of Eq. (6.5), the inequality (6.13) is also a necessary condition of separability. Therefore, a TMGS whose CM has the standard form II is a unique state for which separability reduces to classicality. By examining the sign of the EPR-like correlation function $\tilde{f} := f(\tilde{u}_1, \tilde{u}_2)$, one can check whether the standard-form-II TMGS is classical or not, i.e., whether it possesses or not a well-behaved Glauber-Sudarshan P representation.

To sum up, our optimization method has exploited the EPR-like correlation function (6.4) leading to the separability indicator \tilde{f} , Eq. (6.14). It is worth stressing that this indicator differs in two respects from the previous ones which are specified by Eqs. (4.19), (4.44), and (5.13). First, it has been identified without making any reference to the partial transposition of the density matrix describing a TMGS $\hat{\rho}$. Second, for the time being, one does not know an explicit solution $\{\tilde{u}_1, \tilde{u}_2\}$ of Eqs. (6.9) and (6.11), except for some special classes of TMGSs, such as the thermal states (TSs), the mode-mixed thermal states (MTSs), and the STSs [29, 30], the symmetric ones [29], as well as the states subject to the constraint $\tilde{f} = 0$. By contrast to the three indicators quoted above, we have not established yet in generality that the existing stationary point is unique or that it corresponds to a minimum.

In principle, the EPR-like separability criterion (6.13) is as important as Simon's condition of separability (3.24) that relies on partial transposition. Nevertheless, it suffers from the drawback that, in the general case, it cannot be handled analytically. This makes Simon's PPT criterion of separability (3.24) to prevail in practice. How-

ever, the next theorem explicitly connects the EPR-like inequality (6.13) to the PPT one (3.24).

Theorem 5. *The EPR-like separability condition (6.13) and Simon's PPT one (3.24) are fully and manifestly equivalent.*

Proof. We introduce the following function, which is symmetric in the mode indices 1 and 2:

$$Z(u_1, u_2) := \frac{1}{2} \left\{ \left[\left(b_1 u_1 - \frac{1}{2} \right) \left(b_2 u_2 - \frac{1}{2} \right) - c^2 u_1 u_2 \right] \right. \\ \left. \left[\left(\frac{b_1}{u_1} + \frac{1}{2} \right) \left(\frac{b_2}{u_2} + \frac{1}{2} \right) - \frac{d^2}{u_1 u_2} \right] + \left[\left(b_1 u_1 + \frac{1}{2} \right) \left(b_2 u_2 + \frac{1}{2} \right) - c^2 u_1 u_2 \right] \right. \\ \left. \times \left[\left(\frac{b_1}{u_1} - \frac{1}{2} \right) \left(\frac{b_2}{u_2} - \frac{1}{2} \right) - \frac{d^2}{u_1 u_2} \right] \right\}. \quad (6.17)$$

The expression on the r. h. s. of Eq. (6.17) can be cast into a simpler form:

$$Z(u_1, u_2) = H(d)\mathcal{D} + H(-d)\mathcal{D}^{\text{PT}} + \frac{1}{4u_1 u_2} \Phi(u_1, u_2), \quad (6.18)$$

where $H(x) := \frac{1}{2}[1 + \text{sgn}(x)]$ denotes the Heaviside step function and $\Phi(u_1, u_2)$ is the function (C5). In view of Eqs. (3.16) and (3.24) we have

$$H(d)\mathcal{D} + H(-d)\mathcal{D}^{\text{PT}} = (b_1 b_2 - c^2) (b_1 b_2 - d^2) - \frac{1}{4} (b_1^2 + b_2^2 + 2c|d|) + \frac{1}{16}. \quad (6.19)$$

Let us equate the expressions (6.17) and (6.18) of the value $Z(\tilde{u}_1, \tilde{u}_2)$ corresponding to a standard form II $\mathcal{V}(\tilde{u}_1, \tilde{u}_2)$ of the CM. Taking account of Eqs. (6.14), (6.15), and (C9), we get the identity:

$$H(d)\mathcal{D} + H(-d)\mathcal{D}^{\text{PT}} = \frac{1}{32} \tilde{f} \left\{ \tilde{f}' \left[\left(\frac{b_1}{\tilde{u}_1} + \frac{1}{2} \right) \left(\frac{b_2}{\tilde{u}_2} + \frac{1}{2} \right) - \frac{d^2}{\tilde{u}_1 \tilde{u}_2} \right] + \tilde{f}'' \left[\left(b_1 \tilde{u}_1 + \frac{1}{2} \right) \left(b_2 \tilde{u}_2 + \frac{1}{2} \right) - c^2 \tilde{u}_1 \tilde{u}_2 \right] \right\}. \quad (6.20)$$

In Eq. (6.20), the expression in curly brackets is strictly positive and $\mathcal{D} \geq 0$ for any TMGS $\hat{\rho}$. Accordingly, the separability indicator \tilde{f} and the local symplectic invariant on the l. h. s. do have the same sign. There are two cases:

- $d \geq 0$. Then $\tilde{f} \geq 0$ and $\mathcal{D}^{\text{PT}} \geq 0$: $\hat{\rho}$ separable;
- $d < 0$.

Either $\tilde{f} \geq 0 \iff \mathcal{D}^{\text{PT}} \geq 0$: $\hat{\rho}$ separable,

or $\tilde{f} < 0 \iff \mathcal{D}^{\text{PT}} < 0$: $\hat{\rho}$ entangled.

As the signs of the separability indicators \tilde{f} and \mathcal{D}^{PT} coincide, we conclude they are equivalent in detecting

separability of TMGSs:

$$\begin{aligned} \tilde{f} \geq 0 &\iff \mathcal{D}^{\text{PT}} \geq 0 : \quad \hat{\rho} \text{ separable}; \\ \tilde{f} < 0 &\iff \mathcal{D}^{\text{PT}} < 0 : \quad \hat{\rho} \text{ entangled}. \end{aligned} \quad (6.21)$$

The proof is complete.

It is instructive to present concisely the above-mentioned special classes of TMGSs whose standard-form-II scaling factors are explicitly evaluated.

1. *TMGSs with $c = |d|$:* TSs, MTSSs, and STSSs. The solution

$$\tilde{u}_1 = \tilde{u}_2 = 1 \quad (6.22)$$

of Eqs. (6.9) and (6.11) is specific for TSs ($c = d = 0$), MTSSs ($c = d > 0$), and STSSs ($c = -d > 0$). We have proven its uniqueness. It gives the minimum value (6.7)

$$\tilde{\alpha}_m^2 = \sqrt{\frac{b_2 - \frac{1}{2}}{b_1 - \frac{1}{2}}}.$$

Equations (6.14) and (6.15) give the separability indicator

$$\tilde{f} = 4 \left[\sqrt{\left(b_1 - \frac{1}{2} \right) \left(b_2 - \frac{1}{2} \right)} - c \right], \quad (6.23)$$

and, respectively, the function

$$\tilde{f}' = \tilde{f}'' = 4 \left[\sqrt{\left(b_1 - \frac{1}{2} \right) \left(b_2 - \frac{1}{2} \right)} + c \right] \geq 0. \quad (6.24)$$

Equation (6.23) becomes insightful when written in terms of the symplectic eigenvalues κ_{\pm} for an MTS and κ_{\pm}^{PT} for an STS:

$$\tilde{f} = \tilde{f}_{\text{MT}} = \frac{4^2}{\tilde{f}'} \left(\kappa_- - \frac{1}{2} \right) \left(\kappa_+ - \frac{1}{2} \right) \geq 0, \quad (d > 0), \quad (6.25)$$

$$\tilde{f} = \tilde{f}_{\text{ST}} = \frac{4^2}{\tilde{f}'} \left(\kappa_-^{\text{PT}} - \frac{1}{2} \right) \left(\kappa_+^{\text{PT}} - \frac{1}{2} \right), \quad (d < 0). \quad (6.26)$$

2. *Symmetric TMGSs:* $b_1 = b_2 =: b$. The solution

$$\tilde{u}_1 = \tilde{u}_2 = \sqrt{\frac{b - |d|}{b - c}} \quad (6.27)$$

is specific for symmetric TMGSs, provided that $c > |d|$. The minimum value (6.7) of the parameter α is $\tilde{\alpha}_m = 1$. We have proven the uniqueness of the solution (6.27), as well as its property of being a minimum point of the function (6.8) written with

$b_1 = b_2 =: b$. The corresponding absolute minimum is the EPR-like separability indicator

$$\tilde{f} = 4 \left[\sqrt{(b-c)(b-|d|)} - \frac{1}{2} \right]. \quad (6.28)$$

According to Eqs. (3.21) and (A5),

$$\kappa_- = \sqrt{(b-c)(b-d)}, \quad \kappa_-^{\text{PT}} = \sqrt{(b-c)(b+d)}, \quad (6.29)$$

so that the following relations hold:

$$\tilde{f} = 4 \left(\kappa_- - \frac{1}{2} \right) \geq 0, \quad (d \geq 0), \quad (6.30)$$

$$\tilde{f} = 4 \left(\kappa_-^{\text{PT}} - \frac{1}{2} \right), \quad (d < 0). \quad (6.31)$$

3. *TMGSs at the boundary $\tilde{f} = 0$.* According to Eq. (6.20),

$$\tilde{f} = 0 \iff H(d)\mathcal{D} + H(-d)\mathcal{D}^{\text{PT}} = 0. \quad (6.32)$$

The property $\tilde{f} = 0$ is therefore specific to all TMGSs with $d \geq 0$ at the physicality edge ($\mathcal{D} = 0$), as well as to all those with $d < 0$ at the separability threshold ($\mathcal{D}^{\text{PT}} = 0$). For both limit situations, we find a unique solution of the algebraic system (6.9) and (6.11):

$$\tilde{u}_1 = 2 \frac{c(b_1 b_2 - d^2) + \frac{1}{4}|d|}{b_1|d| + b_2 c}, \quad \tilde{u}_2 = 2 \frac{c(b_1 b_2 - d^2) + \frac{1}{4}|d|}{b_1 c + b_2|d|}. \quad (6.33)$$

Let us focus on the expressions (6.25), (6.26), and (6.30)- (6.31) of the EPR-like correlation function \tilde{f} for MTSs, STSs, and symmetric TMGSs, respectively. These important special GSs illustrate both cases: $d \geq 0$ and $d < 0$. The above explicit formulae clearly display the equivalence between the EPR-like separability condition (6.13) and the PPT one, Eq. (3.24). Such explicit expressions are not available for an arbitrary TMGS. Anyway, in order to decide whether a given TMGS is separable or not, then, by virtue of Eq. (6.21), one is entitled to employ Simon's condition of separability, $\mathcal{D}^{\text{PT}} \geq 0$, instead of the less efficient formula $\tilde{f} \geq 0$.

VII. SUMMARY AND CONCLUSIONS

This work is devoted to an explicit application of EPR-like correlations in detecting Gaussian entanglement. Let us emphasize its most significant achievements. First, we have tackled three correlation functions built with variances of two EPR-like observables in a TMGS. They are a normalized product, a normalized sum, and a non-normalized, but regularized sum. Our main results are

Theorems 1, 2, and 3, which express their absolute minima in terms of either the smaller symplectic eigenvalue κ_-^{PT} or the Simon symplectic invariant \mathcal{D}^{PT} . These lower bounds are explicitly written in Eqs. (4.19), (4.44), and (5.13). All the three minima are EPR-like separability indicators due to Simon's PPT criterion. We have exploited these analytic results to point out that three distinct EPR-like necessary conditions of separability for a TMGS imply the corresponding Peres-Simon PPT condition, Eq. (4.50).

Second, the present development provides a fresh look at the work of Duan *et al.*, which is based on EPR-like observables [14]. Among the original ideas put forward in Ref. [14], the central one is the existence of a standard form II of the CM, which is confirmed here. This CM describes a privileged TMGS whose classicality is equivalent to the separability of the whole set of TMGSs connected to it by local unitary transformations. Thus, the separability properties of the whole class of TMGSs having a given set of standard-form parameters are assigned to this standard-form-II state. The resulting separability criterion is quite special because it is independent of the PPT condition. However, in spite of its soundness, the original EPR-like approach [14] cannot decide if a TMGS is separable or not, except for the special cases discussed in Sec. VI. This happens because it does not provide a general analytic solution. In the present work, the EPR-like correlation function introduced in Ref. [14] is used in a regularized sum form. Its optimal value over the variables α, u_1, u_2 , denoted \tilde{f} , turns out to be a separability indicator for TMGSs. This is obtained as a marker of classicality for the standard-form-II TMGS.

Third, we have explicitly proven that the EPR-like indicator of separability \tilde{f} is equivalent to Simon's PPT separability marker \mathcal{D}^{PT} . We stress that the resulting formula is the first direct connection between two distinct approaches to the separability problem for TMGSs. It might stimulate once more a reflection on the central role of the uncertainty relations in quantum mechanics. Needless to say, the importance of the EPR-like approach is enhanced by our simple proof of its manifest consistency with Simon's PPT separability condition, whose practical usefulness is universally acknowledged.

Appendix A: Symplectic eigenvalues of the covariance matrix \mathcal{V}^{PT}

We focus on the positive definite CM \mathcal{V}^{PT} which is built with the Gaussian operator $\hat{\rho}^{\text{PT}}$ obtained from the TMGS $\hat{\rho}$ by partial transposition of the density matrix. Let us denote its symplectic eigenvalues by κ_{\pm}^{PT} and write down the counterparts of Eqs. (3.18) and (3.19):

$$\det(\mathcal{V}) = (\kappa_+^{\text{PT}})^2 (\kappa_-^{\text{PT}})^2, \quad (A1)$$

$$\mathcal{D}^{\text{PT}} = \left[(\kappa_+^{\text{PT}})^2 - \frac{1}{4} \right] \left[(\kappa_-^{\text{PT}})^2 - \frac{1}{4} \right], \quad (A2)$$

with

$$\kappa_+^{\text{PT}} \geq \kappa_-^{\text{PT}} > 0. \quad (\text{A3})$$

from Eqs. (A1), (A2), and (3.24) we get the biquadratic equation

$$(\kappa^{\text{PT}})^4 - (b_1^2 + b_2^2 - 2cd)(\kappa^{\text{PT}})^2 + \det(\mathcal{V}) = 0, \quad (\text{A4})$$

satisfied by the symplectic eigenvalues:

$$\begin{aligned} (\kappa_{\pm}^{\text{PT}})^2 &= \frac{1}{2} \left[(b_1^2 + b_2^2 - 2cd) \pm \sqrt{\Delta^{\text{PT}}} \right], \\ \Delta^{\text{PT}} &:= (b_1^2 + b_2^2 - 2cd)^2 - 4 \det(\mathcal{V}) \\ &= (b_1^2 - b_2^2)^2 + 4(b_1c - b_2d)(b_2c - b_1d) \geq 0. \end{aligned} \quad (\text{A5})$$

Taking account of Eqs. (A2) and (A3), Simon's separability condition (3.24) states the following alternative:

- $\mathcal{D}^{\text{PT}} \geq 0 \iff \kappa_-^{\text{PT}} \geq \frac{1}{2}$ for separable TMGSs;
- $\mathcal{D}^{\text{PT}} < 0 \iff \kappa_-^{\text{PT}} < \frac{1}{2}$ for entangled TMGSs.

Note that the partial transpose $\hat{\rho}^{\text{PT}}$ of any entangled TMGS $\hat{\rho}$ is no longer a state.

Multiplication of Eq. (A4) by the positive quantities $b_1b_2 - c^2$ and $b_1b_2 - d^2$ from Eq. (3.15) yields two useful identities:

$$\begin{aligned} &\left[b_1(b_1b_2 - c^2) - b_2(\kappa_{\pm}^{\text{PT}})^2 \right] \left[b_2(b_1b_2 - c^2) - b_1(\kappa_{\pm}^{\text{PT}})^2 \right] \\ &= \left[c(\kappa_{\pm}^{\text{PT}})^2 - d(b_1b_2 - c^2) \right]^2; \end{aligned} \quad (\text{A6})$$

$$\begin{aligned} &\left[b_1(b_1b_2 - d^2) - b_2(\kappa_{\pm}^{\text{PT}})^2 \right] \left[b_2(b_1b_2 - d^2) - b_1(\kappa_{\pm}^{\text{PT}})^2 \right] \\ &= \left[c(b_1b_2 - d^2) - d(\kappa_{\pm}^{\text{PT}})^2 \right]^2. \end{aligned} \quad (\text{A7})$$

Equations (A6) and (A7) transform into each other by interchanging the parameters c and $-d$. Another pair of identities related in the same way is obtained when we multiply Eq. (A4) by the product b_1b_2 :

$$\begin{aligned} &\left[b_1(b_1b_2 - c^2) - b_2(\kappa_{\pm}^{\text{PT}})^2 \right] \left[b_2(b_1b_2 - d^2) - b_1(\kappa_{\pm}^{\text{PT}})^2 \right] \\ &= (b_1c - b_2d)^2 (\kappa_{\pm}^{\text{PT}})^2; \end{aligned} \quad (\text{A8})$$

$$\begin{aligned} &\left[b_1(b_1b_2 - d^2) - b_2(\kappa_{\pm}^{\text{PT}})^2 \right] \left[b_2(b_1b_2 - c^2) - b_1(\kappa_{\pm}^{\text{PT}})^2 \right] \\ &= (b_2c - b_1d)^2 (\kappa_{\pm}^{\text{PT}})^2. \end{aligned} \quad (\text{A9})$$

We finally mention an identity involving the determinant (A2):

$$\begin{aligned} &4 \left[b_2(b_1b_2 - c^2) - \frac{1}{4}b_1 \right] \left[b_2(b_1b_2 - d^2) - \frac{1}{4}b_1 \right] \\ &- \left(b_2^2 - \frac{1}{4} - cd \right)^2 = 4 \left(b_2^2 - \frac{1}{4} \right) \mathcal{D}^{\text{PT}}. \end{aligned} \quad (\text{A10})$$

Appendix B: Hessian matrices

1. The function $\ln[E(\lambda, \mu)]$

By using Eq. (4.11), we evaluate the second-order derivatives of the function $\ln[E(\lambda, \mu)]$ at the stationary point (4.17):

$$\begin{aligned} H_{11} &:= \frac{\partial^2 \ln(E)}{\partial \lambda^2}(\lambda_m, \mu_m), \quad H_{22} := \frac{\partial^2 \ln(E)}{\partial \mu^2}(\lambda_m, \mu_m), \\ H_{12} &:= \frac{\partial^2 \ln(E)}{\partial \lambda \partial \mu}(\lambda_m, \mu_m). \end{aligned} \quad (\text{B1})$$

By use of Eq. (4.16), we get the following entries of the Hessian matrix:

$$\begin{aligned} H_{11} &= \frac{2(b_1b_2 - c^2)}{[\Delta Q(\lambda_m)]^4} > 0, \quad H_{22} = \frac{2(b_1b_2 - d^2)}{[\Delta P(\mu_m)]^4} > 0, \\ H_{12} &= -\frac{2}{(1 + \lambda_m\mu_m)^2} < 0. \end{aligned} \quad (\text{B2})$$

Then, in view of Eq. (A5), the Hessian determinant is positive:

$$\det(H) = \frac{4\sqrt{\Delta^{\text{PT}}}}{(\kappa_-^{\text{PT}})^2(1 + \lambda_m\mu_m)^4} > 0. \quad (\text{B3})$$

Consequently, the Hessian matrix (B1) is positive definite. The absolute minimum of the function $E(\lambda, \mu)$, Eq. (4.11), is therefore

$$E_m = E(\lambda_m, \mu_m). \quad (\text{B4})$$

2. The function $F(\alpha^2, u_1, u_2)$

We have evaluated the Hessian matrix of the function $F(\alpha^2, u_1, u_2)$, Eq. (4.24), at its stationary point, Eqs. (4.41) and (4.42). In the sequel, the indices 1, 2, 3 refer to the independent variables u_1, u_2, α^2 , respectively. Let us write the expressions of three principal minors: the diagonal entry H_{11} , the cofactor A_{33} , and the Hessian determinant $\det(H)$. First, the following expression of H_{11} holds provided that $c + d > 0$:

$$\begin{aligned} H_{11} &= \frac{\sqrt{\gamma}}{2 \left(\alpha_m^2 + \frac{1}{\alpha_m^2} \right) u_{1m}^2 u_{2m}} \\ &\times \left[\left(1 + \frac{1}{u_{1m}^2} \right) \frac{cu_{1m}u_{2m} + d}{1 - \frac{1}{u_{1m}^2}} + 2 \frac{c\gamma + d}{1 - \frac{1}{u_{1m}^2}} \right] > 0. \end{aligned} \quad (\text{B5})$$

A suitable simplification in Eq. (B5) yields a formula which is valid also in the limit case of the two-mode STSs

$(c + d = 0) :$

$$\begin{aligned}
H_{11} &= \frac{\sqrt{\gamma}}{2 \left(\alpha_m^2 + \frac{1}{\alpha_m^2} \right) u_{1m}^2 u_{2m}} \\
&\times \frac{1}{b_1 \sqrt{\Delta^{\text{PT}}} + b_1 (b_1^2 - b_2^2) - 2d (b_1 c - b_2 d)} \\
&\times \left\{ \left(1 + \frac{1}{u_{1m}^2} \right) \frac{b_1 (b_1 b_2 - d^2)}{\gamma (b_1 b_2 - c^2)} [c (b_1^2 + b_2^2) - d (2b_1 b_2) \right. \\
&\left. + c \sqrt{\Delta^{\text{PT}}} \right] + 4 (b_1 b_2 - d^2) (b_1 c - b_2 d) \Big\} > 0. \quad (\text{B6})
\end{aligned}$$

Second, we present a general symmetric expression of the cofactor A_{33} :

$$\begin{aligned}
A_{33} &= \frac{1}{\left(\alpha_m^2 + \frac{1}{\alpha_m^2} \right)^2 u_{1m} u_{2m}} \left\{ b_1 b_2 \left(\frac{1}{u_{2m}} - \frac{1}{u_{1m}} \right)^2 \right. \\
&+ \frac{4 (b_1 b_2 - c^2) (b_1 c - b_2 d) (b_2 c - b_1 d)}{c (b_1^2 + b_2^2) - d (2b_1 b_2) + c \sqrt{\Delta^{\text{PT}}}} \\
&\times \frac{1}{c - d} \left(1 + \frac{1}{u_{1m} u_{2m}} \right) \Big\} > 0. \quad (\text{B7})
\end{aligned}$$

Third, we have obtained the Hessian determinant:

$$\begin{aligned}
\det(H) &= \frac{4}{\alpha_m^4 \left(\alpha_m^2 + \frac{1}{\alpha_m^2} \right)^3 (u_{1m} u_{2m})^{\frac{3}{2}}} \\
&\times \frac{4 (b_1 b_2 - c^2) (b_1 c - b_2 d) (b_2 c - b_1 d)}{c (b_1^2 + b_2^2) - d (2b_1 b_2) + c \sqrt{\Delta^{\text{PT}}}} > 0. \quad (\text{B8})
\end{aligned}$$

The obvious inequalities (B6)- (B8) show that the Hessian matrix under discussion is positive definite. Therefore, the unique stationary point $\{\alpha_m^2, u_{1m}, u_{2m}\}$ is a minimum point where the function $F(\alpha^2, u_1, u_2)$ reaches its absolute minimum (4.43).

Needless to say, Eqs. (B6)- (B8) considerably simplify in the particular cases of STSs and symmetric states. We list the resulting formulae as follows.

1. Two-mode STSs:

$$\begin{aligned}
H_{11} &= \frac{1}{\sqrt{\delta}} \left\{ b_1 \sqrt{\delta} - [b_1 (b_1 - b_2) + c^2] \right\} > 0; \\
A_{33} &= \frac{c^2}{\delta} (b_1 + b_2) [(b_1 + b_2) - \sqrt{\delta}] > 0; \\
\det(H) &= \delta^{-\frac{3}{2}} c^2 (b_1 + b_2) [\sqrt{\delta} + (b_1 - b_2)]^2 \\
&\times [(b_1 + b_2) - \sqrt{\delta}] > 0. \quad (\text{B9})
\end{aligned}$$

2. Symmetric TMGSs:

$$\begin{aligned}
H_{11} &= \left(\frac{b-c}{b+d} \right)^{\frac{1}{2}} \frac{1}{b+d} \{ b [(b-c) + (b+d)] \\
&+ 2(b-c)(b+d) \} > 0; \\
A_{33} &= \frac{1}{2} \left(\frac{b-c}{b+d} \right)^2 b [(b-c) + (b+d)] > 0; \\
\det(H) &= \left(\frac{b-c}{b+d} \right)^{\frac{3}{2}} b(b-c)(c-d) > 0. \quad (\text{B10})
\end{aligned}$$

The above formulae have been checked by direct evaluation of the corresponding Hessian matrices starting from Eqs. (4.46) and (4.47), respectively.

3. The function $G(\xi, \eta, u_1)$

Let us assign the indices 1, 2, 3 to the independent variables u_1, ξ, η , respectively. The Hessian matrix of the function $G(\xi, \eta, u_1)$, Eq. (5.7), evaluated at the stationary point (5.10) has the following entries:

$$\begin{aligned}
H_{11} &= \frac{1}{(b_2^2 - \frac{1}{4}) u_{1m}^3} \left\{ \left[b_2 (b_1 b_2 - d^2) - \frac{1}{4} b_1 \right] \right. \\
&+ \frac{1}{2} \left[b_2 (b_1 b_2 - c^2) - \frac{1}{4} b_1 \right] (u_{1m}^2 + 1) \\
&+ \frac{1}{2} b_2 c^2 (u_{1m} - 1)^2 + \frac{1}{2} c (2b_2 c + d) u_{1m} \Big\} > 0, \\
H_{22} &= 2b_2, \quad H_{33} = 2b_2, \\
H_{12} &= -c u_{1m}^{-\frac{1}{2}}, \quad H_{13} = -d u_{1m}^{-\frac{3}{2}}, \quad H_{23} = -1. \quad (\text{B11})
\end{aligned}$$

Three nested principal minors are clearly positive: the diagonal entry H_{33} , the cofactor A_{11} , and the Hessian determinant $\det(H)$. Indeed,

$$\begin{aligned}
H_{33} &= 2b_2 > 0, \quad A_{11} = 4 \left(b_2^2 - \frac{1}{4} \right) > 0, \\
\det(H) &= \frac{2}{u_{1m}^3} \left\{ 2 \left[b_2 (b_1 b_2 - d^2) - \frac{1}{4} b_1 \right] \right. \\
&+ \left[b_2 (b_1 b_2 - c^2) - \frac{1}{4} b_1 \right] (u_{1m}^2 + 1) \\
&+ b_2 (c^2 - d^2) \Big\} > 0. \quad (\text{B12})
\end{aligned}$$

Accordingly, the Hessian matrix (B11) is positive definite. This means that the unique stationary point $\{\xi_m, \eta_m, u_{1m}\}$, Eq. (5.10), is a minimum point where the function $G(\xi, \eta, u_1)$ reaches its absolute minimum (5.11).

Appendix C: Existence of a standard form II $\mathcal{V}(\tilde{u}_1, \tilde{u}_2)$ of the covariance matrix

We prove Theorem 4, stated in Sec. VI, which asserts the existence of a solution of Eqs. (6.9) and (6.11) for any TMGS in the classicality range of its scaling factors.

Proof. Note that Eq. (6.9) displays the pairings:

$$u_1 = 1 \iff u_2 = 1, \quad u_1 = 2b_1 \iff u_2 = 2b_2. \quad (\text{C1})$$

As a matter of fact, Eq. (6.9) is a quadratic one in each of the variables u_1 and u_2 , leading to a bijective continuous function,

$$u_2 = h(u_1), \quad h : [1, 2b_1] \longrightarrow [1, 2b_2], \quad (\text{C2})$$

which reads:

$$h(u_1) = \frac{2b_2 u_1 (2b_1 u_1 - 1)}{b_1 (u_1^2 - 1) + \sqrt{\Delta_1}},$$

$$\Delta_1 := b_1^2 (u_1^2 - 1)^2 + 4b_2^2 u_1 (2b_1 - u_1) (2b_1 u_1 - 1). \quad (\text{C3})$$

One gets the inverse function

$$u_1 = h^{-1}(u_2), \quad h^{-1} : [1, 2b_2] \longrightarrow [1, 2b_1], \quad (\text{C4})$$

by interchanging the mode indices 1 and 2 in Eq. (C3). The bijective function $h(u_1)$, Eq. (C3), is strictly and continuously increasing from the initial value $h(1) = 1$ to the final one $h(2b_1) = 2b_2$. Remark that for a symmetric TMGS it reduces to the identity function:

$$u_2 = u_1, \quad (b_1 = b_2 =: b).$$

Coming back to the general case, we introduce a function of two variables suggested by Eq. (6.11):

$$\Phi(u_1, u_2) := b_1 b_2 (u_1^2 - 1) (u_2^2 - 1) - (c u_1 u_2 - |d|)^2. \quad (\text{C5})$$

The existence of a solution of the algebraic system (6.9) and (6.11) is equivalent to that of a zero of the one-variable function

$$\phi(u_1) := \Phi(u_1, h(u_1)), \quad (u_1 \in [1, 2b_1]). \quad (\text{C6})$$

In order to check if such a zero exists indeed, we have to examine the sign of the values of the function (C6) at the end points of its domain:

$$\phi(1) = \Phi(1, 1) = -(c - |d|)^2 : \quad \phi(1) \leq 0; \quad (\text{C7})$$

$$\begin{aligned} \phi(2b_1) &= \Phi(2b_1, 2b_2) \\ &= 16b_1 b_2 \left[\mathcal{D} + d^2 \left(b_1 b_2 - c^2 - \frac{1}{4} \right) + \frac{1}{2} c(|d| + d) \right] \\ &\quad + 4d^2 \left(b_1 b_2 - \frac{1}{4} \right) : \quad \phi(2b_1) \geq 0. \end{aligned} \quad (\text{C8})$$

The inequality in Eq. (C8) stems from conditions (3.14)–(3.16), which are equivalent to the Robertson-Schrödinger uncertainty relation (3.4). Therefore, by virtue of continuity, the function (C6) has at least a zero in the classicality interval $u_1 \in [1, 2b_1]$ for any TMGS. We denote $\{\tilde{u}_1, \tilde{u}_2 = h(\tilde{u}_1)\}$ the corresponding solution of Eqs. (6.9) and (6.11), so that

$$\phi(\tilde{u}_1) = \Phi(\tilde{u}_1, \tilde{u}_2) = 0, \quad (\tilde{u}_1 \in [1, 2b_1], \tilde{u}_2 \in [1, 2b_2]). \quad (\text{C9})$$

This concludes the proof.

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